Exactly Solvable One-Dimensional Inhomogeneous Models

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We present a simple way of constructing one-dimensional inhomogeneous models (random or quasiperiodic) which can be solved exactly. We treat the example of an Ising chain in a varying magnetic field, but our procedure can easily be extended to other one-dimensional inhomogeneous models. For all the models we can construct, the free energy and its derivatives with respect to temperature can be computed exactly at one particular temperature.

1. INTRODUCTION

One-dimensional models in classical statistical mechanics are known to be exactly and easily solvable. However, as soon as the one-dimensional model becomes inhomogeneous,[1] i.e., the parameters that define the model (magnetic field or nearest neighbor interactions for Ising models, potential in the Anderson model, etc.) vary in space (random models or quasiperiodic models), finding the exact solution usually becomes next to impossible except in a few special examples[2] or in some limiting cases[3,4] (zero temperature, weak disorder,...).

In the present work, we show how to construct a large class of inhomogeneous models (random or quasiperiodic) for which the free energy (or Liapunov exponent) can be computed exactly. In order to illustrate our method, we restrict our discussion to an Ising chain with a

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varying external field. Our method can nevertheless be easily extended to Ising chains with nonconstant interactions in a varying field, to the problem of diffusion in inhomogeneous media, to the problem of a quantum particle in a varying potential, etc.

Our approach is rather artificial because we start by giving the solution and then build the model knowing its solution. As a consequence, we compute the free energy and its derivatives at a specific temperature only. The calculation of the nth derivative with respect to temperature is possible. Yet the complexity increases rapidly with the order of derivation. The knowledge of the free energy at a single point is very reminiscent of some spin models for which the partition function can be computed exactly on some special lines in a phase diagram (the so-called disordered lines)[5] and also of the exact results known on the Nishimori line in spin glasses.[6]

In Section 6 we compare our work to that of Percus,[7] who computes the site-dependent magnetic field that produces a given magnetization profile.

2. SOLVING THE MODEL

Consider an Ising chain in a variable external field h_i. Its Hamiltonian is

$$\mathscr{H}(\sigma) = -\sum_{i=1}^{N-1} J\sigma_i\sigma_{i+1} - \sum_{i=1}^{N} h_i\sigma_i$$

where as usual σ_i denotes the Ising spin ($\sigma_i = \pm 1$) at site i and J is the coupling constant. For certain choices of the h_i we shall see that we can compute the free energy exactly.

Let

$$Z_N = \sum_{\sigma \in \{-1,1\}^N} \exp[-\mathscr{H}(\sigma)]$$

be the partition function (notice we have normalized the temperature $T=1$). The easiest way to calculate Z_N is to use the transfer matrix technique, which consists in expressing Z_{n+1} in terms of Z_n. To be more specific, define $Z_n(+)$ to be the partition function of the Ising chain starting at site 1, ending at site n, with the extra condition $\sigma_n = +1$. In the same fashion, we define $Z_n(-)$ with $\sigma_n = -1$. Hence, for $\varepsilon = \pm$,

$$Z_n(\varepsilon) = \sum_{\sigma \in \{-1,1\}^{n-1}} \exp\left(J\sum_{i=1}^{n-2} \sigma_i\sigma_{i+1} + J\varepsilon\sigma_{n-1} + \sum_{i=1}^{n-1} h_i\sigma_i + \varepsilon h_n\right)$$

It is easy to see that

$$Z_{n+1}(+) = e^{h_{n+1}}[e^J Z_n(+) + e^{-J} Z_n(-)]$$
$$Z_{n+1}(-) = e^{-h_{n+1}}[e^{-J} Z_n(+) + e^J Z_n(-)]$$

Put

$$r_n = Z_n(+)/Z_n(-) \tag{1}$$

The above recurrence yields

$$r_{n+1} = e^{2h_{n+1}} \frac{e^{2J} r_n + 1}{r_n + e^{2J}} \tag{2}$$

Usually one is asked to compute the r_n, given the h_n. Indeed, the free energy ψ per spin is expressed in terms of the r_n in the following way:

$$-\psi = \lim_{N \to \infty} \frac{1}{N} \log Z_N$$

$$= \lim_{N \to \infty} \frac{1}{N} \log[Z_N(+) + Z_N(-)]$$

$$= \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=2}^{N} \log \left(\frac{Z_n(+) + Z_n(-)}{Z_{n-1}(+) + Z_{n-1}(-)} \right) + \frac{1}{N} \log[Z_1(+) + Z_1(-)] \right\}$$

$$= \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=2}^{N} \log[e^{-h_n}(e^{-J} r_{n-1} + e^J)] + \frac{1}{N} \log \frac{r_N + 1}{r_1 + 1} \right\} \tag{3a}$$

We now make the assumption that as N increases to infinity, r_N is bounded. This is of course the common case. Then, using (2), we have

$$-\psi = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log[(e^{-J} r_{n-1} + e^J) e^{-h_n}]$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{2} \log \frac{(e^J + e^{-J} r_{n-1})(e^{-J} + e^J r_{n-1})}{r_n}$$

$$= \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log \frac{1 + 2r_{n-1} \cosh 2J + r_{n-1}^2}{r_n}$$

Notice that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log \frac{1}{r_n} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=2}^{N} \log \frac{1}{r_{n-1}}$$

Thence

$$-\psi = \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log \left(r_n + \frac{1}{r_n} + 2 \cosh 2J \right) \qquad (3b)$$

This calculation is valid provided the limits exist.

In the usual approach, it is difficult to compute the free energy because the sequence r_n is defined through the recurrence relation (2). If h_n is not constant, r_n can be too complicated to allow the calculation of the free energy using (3b). We propose here to solve the problem from the "other end." We are given the sequence r_n. We then obtain both the external field h_n from formula (2)

$$e^{2h_n} = r_n \frac{r_{n-1} + e^{2J}}{e^{2J} r_{n-1} + 1} \qquad (4)$$

and the free energy ψ from formula (3b). We illustrate this principle in the next Section, leaving the discussion of the derivatives of ψ to Section 5.

3. A GENERIC EXAMPLE

Let us first recall a useful result due to H. Weyl (see Ref. 8).

Lemma.   If (u_n) is an equidistributed sequence (mod 1), then for all Riemann integrable functions Φ with period 1

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \Phi(u_n) = \int_0^1 \Phi(x)\, dx$$

Examples of such sequences are $u_n = n\alpha$ (α irrational), $u_n = n^2\alpha$ (α irrational), $u_n = \sqrt{n}$, etc., As for Riemann integrable functions, any real periodic continuous functions with period 1 will suffice for our applications.

Suppose we are given an equidistributed sequence (mod 1), say u_n, and a continuous function φ with period 1. We choose $r_n = \varphi(u_n)$. In order to give a physical meaning to r_n, we must assume $\varphi > 0$ [because r_n is the ratio of two positive numbers; Eq. (1)]. We shall actually demand slightly more, namely that there exists a constant $C > 0$ such that for all $x \in (0, 1)$, $\varphi(x) > C$. Apply Weyl's lemma to

$$\Phi(x) = \log \left[\varphi(x) + \frac{1}{\varphi(x)} + 2 \cosh 2J \right]$$

Then the free energy is given by

$$-\psi = \frac{1}{2} \int_0^1 \log \left[\varphi(x) + \frac{1}{\varphi(x)} + 2 \cosh 2J \right] dx \qquad (5)$$

4. QUASIPERIODIC AND RANDOM EXAMPLES

One can choose for u_n a quasiperiodic sequence, say $u_n = (n+1)\alpha$, where α is irrational. For example, take

$$r_n = \lambda_n + (\lambda_n^2 - 1)^{1/2} = \varphi(u_n)$$

where

$$\lambda_n = A + B \cos 2\pi(n+1)\alpha, \qquad 0 < B < A - 1, \quad \alpha \text{ irrational}$$

The sequence r_n is quasiperiodic.

The field h_n is then given by (4). More specifically

$$
\begin{aligned}
h_n = &\tfrac{1}{2}[\log(A + B \cos 2\pi(n+1)\alpha + \{[A + B \cos 2\pi(n+1)\alpha]^2 - 1\}^{1/2}) \\
&+ \log\{e^{2J} + A + B \cos 2\pi n\alpha + [(A + B \cos 2\pi n\alpha)^2 - 1]^{1/2}\} \\
&- \log\{1 + Ae^{2J} + Be^{2J} \cos 2\pi n\alpha \\
&+ e^{2J}[(A + B \cos 2\pi n\alpha)^2 - 1]^{1/2}\}]
\end{aligned}
\tag{6}
$$

The field h_n is seen to be also quasiperiodic. A simple calculation using formula (5) leads to

$$-\psi = \tfrac{1}{2} \log\{A + \cosh 2J + [(A + \cosh 2J)^2 - B^2]^{1/2}\} \tag{7}$$

We thus conclude that for a quasiperiodic field h_n given by (6), the free energy can be computed exactly and the result is (7).

In the previous example, r_n and h_n are deterministic sequences. In the same fashion one can construct random sequences. For example, if we choose

$$\lambda_n = A + B \cos(2\pi X_n)$$

where X_n is a uniformly distributed random variable on the interval $(0, 1)$, then the field h_n becomes random and is given by

$$
\begin{aligned}
h_n = &\tfrac{1}{2}(\log\{A + B \cos 2\pi X_n + [(A + B \cos 2\pi X_n)^2 - 1]^{1/2}\} \\
&+ \log\{e^{2J} + A + B \cos 2\pi X_{n-1} + [(A + B \cos 2\pi X_{n-1})^2 - 1]^{1/2}\} \\
&- \log\{1 + Ae^{2J} + Be^{2J} \cos 2\pi X_{n-1} \\
&+ e^{2J}[(A + B \cos 2\pi X_{n-1})^2 - 1]^{1/2}\})
\end{aligned}
\tag{8}
$$

The sequence h_n is random, but h_n is correlated to h_{n-1} and to h_{n+1}

because h_n and h_{n-1} both depend on X_{n-1}. Observe, however, that if the X_n are independent, there is no further correlation

$$[\langle h_{n+p} h_n \rangle - \langle h_{n+p} \rangle \langle h_n \rangle = 0 \quad \text{for} \quad p \geqslant 2 \quad \text{or} \quad p \leqslant -2]$$

It is clear that for the random sequence (8), the free energy is again given by (7).

5. DERIVATIVES OF THE FREE ENERGY

We have seen in Sections 2 and 3 that, given the sequence r_n, the sequence h_n is defined by

$$h_n = \frac{1}{2} \log \frac{r_n (r_{n-1} + e^{2J})}{e^{2J} r_{n-1} + 1} \tag{9}$$

and the free energy ψ at $\beta = 1$ is given by

$$-\psi = \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log \left(\frac{1}{r_n} + r_n + 2 \cosh 2J \right) \tag{10}$$

Suppose we want to compute the free energy of the chain where h_n is defined by (9) at temperature $T = 1/\beta \neq 1$. We then have to compute s_n defined by the recurrence

$$s_{n+1} = e^{2\beta h_{n+1}} \frac{e^{2\beta J} s_n + 1}{e^{2\beta J} + s_n} \tag{11}$$

As before, the free energy ψ will be given by

$$-\beta\psi = \frac{1}{2} \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \log \left(\frac{1}{s_n} + s_n + 2 \cosh 2\beta J \right) \tag{12}$$

Obviously for $\beta = 1$, one has $s_n = r_n$. Computing the derivatives of ψ with respect to β at $\beta = 1$ involves computing the derivatives of s_n at $\beta = 1$. Let us show how to calculate $d\psi/d\beta$:

$$-\left. \frac{d(\beta\psi)}{d\beta} \right|_{\beta=1} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left[d(r_n) + c(r_n) \left. \frac{s_n'}{s_n} \right|_{\beta=1} \right] \tag{13}$$

where

$$c(r_n) = \frac{1}{2} \frac{r_n^2 - 1}{r_n^2 + 1 + 2r_n \cosh 2J}$$

$$d(r_n) = \frac{2J r_n \sinh 2J}{r_n^2 + 1 + 2r_n \cosh 2J} \tag{14}$$

The careful reader may well criticize the liberty with which we inverted the limit operator and the derivation. For each choice of the sequence r_n, one should actually justify this step. However, at this stage we wish only to present a method and we disregard this mathematical technicality.

Formula (13) shows that the calculation of $d\psi/d\beta$ requires the calculation of s'_n/s_n at $\beta = 1$ only. Formula (11) implies the following induction formula, which relates s'_{n+1}/s_{n+1} to s'_n/s_n:

$$\frac{s'_{n+1}}{s_{n+1}} = a(r_n)\frac{s'_n}{s_n} + b(r_{n+1}, r_n) \tag{15}$$

where

$$a(r_n) = r_n\left(\frac{e^{2J}}{e^{2J}r_n + 1} - \frac{1}{e^{2J} + r_n}\right) \tag{16}$$

$$b(r_{n+1}, r_n) = 2h_{n+1} + 2J\left(\frac{e^{2J}r_n}{e^{2J}r_n + 1} - \frac{e^{2J}}{r_n + e^{2J}}\right) \tag{17}$$

Since the r_n are positive, it is easily checked that

$$|a(r_n)| \leqslant |\tanh J| < 1 \tag{18}$$

Therefore, one can solve (15) easily:

$$\frac{s'_{n+1}}{s_{n+1}} = b(r_{n+1}, r_n) + \sum_{p=0}^{\infty}\left[\prod_{q=0}^{p} a(r_{n-q})\right]b(r_{n-p}, r_{n-p-1}) \tag{19}$$

Going back to (13), one finds

$$-\frac{d(\beta\psi)}{d\beta}\bigg|_{\beta=1} = X + Y + \sum_{p=1}^{\infty} V_p \tag{20}$$

where

$$X = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} d(r_n)$$

$$Y = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c(r_n)\,b(r_{n+1}, r_n) \tag{21}$$

$$V_p = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} c(r_n)\left[\prod_{q=1}^{p} a(r_{n-q})\right]b(r_{n-p}, r_{n-p-1})$$

Remembering that the sequence r_n is given, it is now left to compute X, Y, and V_p either by Weyl's theorem if r_n is uniformly distributed modulo 1 or

by the strong law of large numbers if the r_n are random variables. More specifically, if $r_n = \varphi(n\alpha)$, α irrational, then

$$X = \int_0^1 d(\varphi(x))\, dx \tag{22}$$

$$Y = \int_0^1 c(\varphi(x))\, b(\varphi(x+\alpha), \varphi(x))\, dx \tag{23}$$

$$V_p = \int_0^1 c(\varphi(x)) \left[\prod_{q=0}^p a(\varphi(x - q\alpha)) \right] b(\varphi(x - p\alpha), \varphi(x - p\alpha - \alpha))\, dx \tag{24}$$

and if r_n are independent random variables uniformly distributed on $(0, 1)$, X is still given by (22), whereas

$$Y = \int_0^1 \int_0^1 c(\varphi(x))\, b(\varphi(y), \varphi(x))\, dx\, dy \tag{25}$$

$$V_p = \left[\int_0^1 c(\varphi(x))\, dx \right] \left[\int_0^1 a(\varphi(x))\, dx \right]^{p-1}$$

$$\times \left[\int_0^1 \int_0^1 a(\varphi(x))\, b(\varphi(y), \varphi(x))\, dx\, dy \right] \tag{26}$$

Notice that in this last case, the series $\sum_p V_p$ is a geometric convergent series.

Higher derivatives could be calculated in the same fashion. It goes without saying that their expression is more complex, because it involves higher derivatives of s_n. This is of course unfortunate. Had we been able to compute the derivatives, Taylor's formula would have given us the expression of the free energy at all temperatures.

The reason the construction of solvable inhomogeneous models is possible is rather obvious. It is clear that for these models, if one changes the temperature or any other physical parameter, the solution described above no longer holds. However, we believe that it is interesting to dispose of a class of models for which the free energy (or the Liapunov exponent) and its derivatives can be known exactly, though it is only known at a single temperature.

It is also clear that by choosing a more complicated sequence r_n one can obtain sequences h_n that may look less bizarre than (6) or (8) (the sequence r_n we used in Section 4 was chosen in order to make the calculation of ψ as simple as possible).

It would be interesting to see whether the very simple argument used here could be generalized to construct inhomogeneous models which would

remain solvable at more than one temperature. A first step in that direction was discussed in the previous section. It would also be interesting to generalize the above argument to transfer matrices of size larger than 2 in order to treat the case of inhomogeneous models on strips.

6. COMPARISON WITH THE METHOD OF PERCUS[7]

The difference between our approach and that of Percus is that we are given the parameters $r_n = Z_n^{(+)}/Z_n^{(-)}$, whereas he is given the more physical quantities $m_n = (\partial/\partial h_n) \log Z_n$, which represent the magnetization at site n (mean spin). We express the physical characteristics of the model in terms of r_n and Percus in terms of m_n.

Now, even though the magnetization m_n has a more physical meaning than r_n, there seems to be some gain in considering r_n. For instance, on the one hand, Percus notices that the external field h_n is a function of the three variables m_{n-1}, m_n, and m_{n+1} [see his equality,[7] p. 303, relating $u(x)$, σ_{x-1}, σ_x, σ_{x+1}]. On the other hand, our Eq. (4) shows that h_n is a function of two variables only, r_{n-1} and r_n. We have reduced the dimension.

The two approaches are nevertheless consistent with one another. To see this, it suffices to prove that r_n is a function of m_n and m_{n+1}. Indeed, recall formulas (2) and (3a):

$$r_{n+1} = e^{2h_{n+1}} \frac{e^{2J}r_n + 1}{r_n + e^{2J}}$$

$$\log Z_N = \sum_{n=2}^{N} \log[e^{-h_n}(e^{-J}r_{n-1} + e^J)] + \log \frac{r_N + 1}{r_1 + 1}$$

We now compute the magnetization,

$$m_n = \frac{\partial}{\partial h_n} \log Z_N$$

If we consider the external field h_n as given, then formula (2) shows that r_n is a function of h_1, h_2,..., h_n. Hence, in the thermodynamic limit

$$m_n = -1 + \sum_{i=0}^{\infty} \frac{1}{r_{n+i} + e^{2J}} \frac{\partial r_{n+i}}{\partial h_n}$$

$$m_{n+1} = -1 + \sum_{i=1}^{\infty} \frac{1}{r_{n+i} + e^{2J}} \frac{\partial r_{n+i}}{\partial h_{n+1}}$$

Put

$$S = \sum_{i=1}^{\infty} \frac{1}{r_{n+i} + e^{2J}} \frac{\partial r_{n+i}}{\partial r_{n+1}}$$

Then

$$m_n = -1 + S \frac{\partial r_{n+1}}{\partial h_n} + \frac{1}{r_n + e^{2J}} \frac{\partial r_n}{\partial h_n}$$

and

$$m_{n+1} = -1 + S \frac{\partial r_{n+1}}{\partial h_{n+1}}$$

We now wish to eliminate S between m_n and m_{n+1}. First note that

$$\partial r_n / \partial h_n = 2r_n$$

and

$$\frac{\partial r_{n+1}}{\partial h_n} = e^{2h_{n+1}} \frac{e^{4J} - 1}{(r_n + e^{2J})^2} = r_{n+1} \frac{e^{4J} - 1}{(r_n + e^{2J})(e^{2J}r_n + 1)}$$

Hence

$$m_{n+1} = -1 + 2r_{n+1}S$$

$$m_n = -1 + 2r_n \left[r_{n+1} \frac{e^{4J} - 1}{(r_n + e^{2J})(e^{2J}r_n + 1)} S + \frac{1}{r_n + e^{2J}} \right]$$

We finally eliminate S:

$$m_n = -1 + 2r_n \left[\frac{1 + m_{n+1}}{2} \frac{e^{4J} - 1}{(r_n + e^{2J})(e^{2J}r_n + 1)} + \frac{1}{r_n + e^{2J}} \right]$$

This rather clumsy expression relates r_n to m_n and m_{n+1} and establishes our claim.

Notice, however, that the expression for r_n in terms of m_n and m_{n+1} involves a second-degree equation, which seems quite unwieldy. The relationship between our point of view and that of Percus, even though clear, appears to be too complicated to allow a mere translation from one system to another.

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